The Drawing Thickness of Graph Drawings

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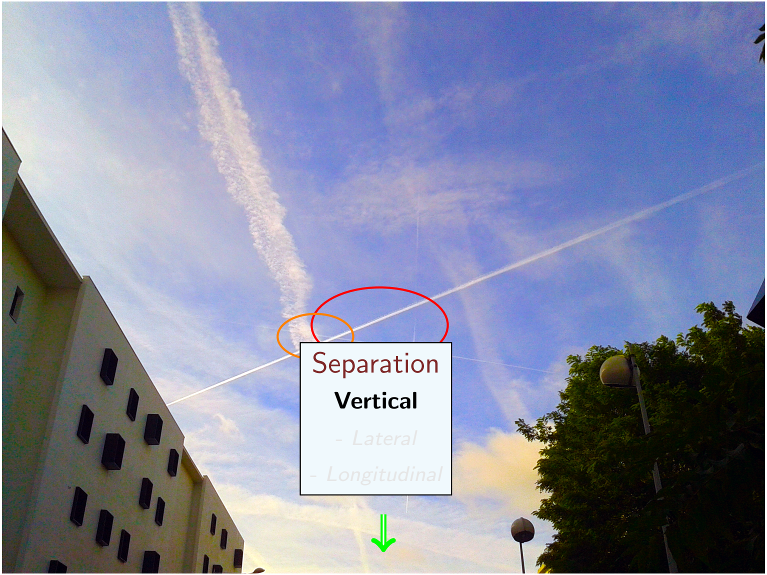
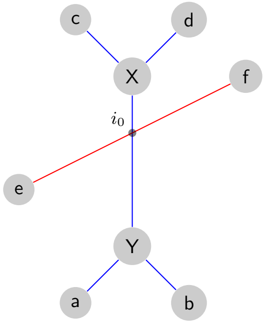
Outline

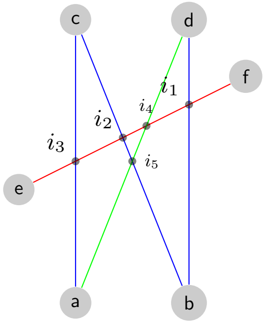
Motivation: Air Traffic Management



Motivation: Air Traffic Management

✚ Maximization of "free flight" airspace

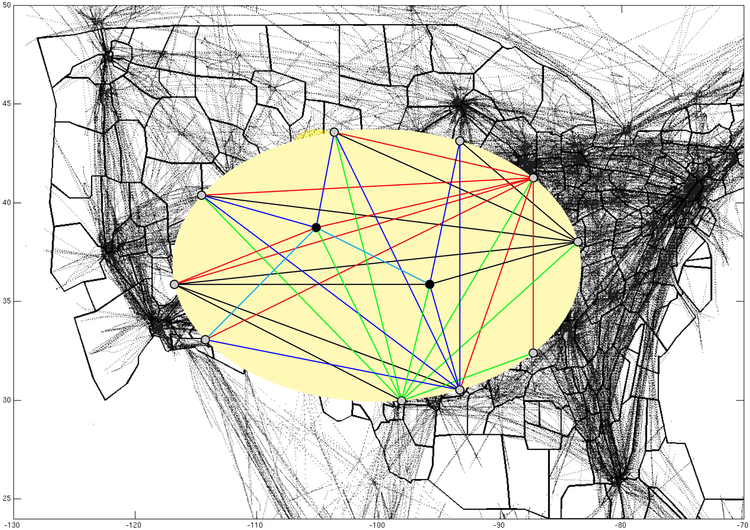




✘ *Direct-to flight (as a choice among "free flight") increases the complexity of air traffic patterns*

Actually... ✔ *Direct-to flight increases the complexity of air traffic patterns and we have something to study...*

Motivation: Air Traffic Management

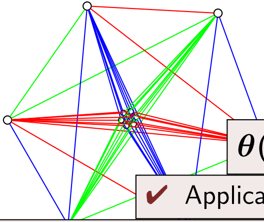


How to model? – Graph drawing & thicknesses

Geometric thickness ($\bar{\theta}$)

Dillencourt et al. (2000)

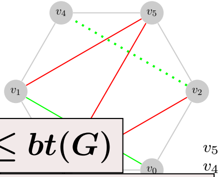
✚ only straight lines

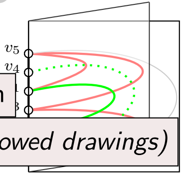


Book thickness (bt)

Bernhart and Kainen (1979)

✚ convex positioning of nodes



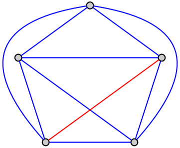


$$\boldsymbol{\theta(G) \leq \bar{\theta}(G) \leq bt(G)}$$

✔ Applications in VLSI & graph visualization

✘ *θ, $\bar{\theta}$, bt characterize the graph (minimizations over all allowed drawings)*

Tutte (1963), "classical" planar decomposition



Geometric graphs and graph drawings

Definition 1.1 (Geometric graph, Bose et al. (2006), many Erdös papers).

A geometric graph G is a pair $(V(G), E(G))$ where $V(G)$ is a set of points in the plane in general position and $E(G)$ is set of closed segments with endpoints in $V(G)$. Elements of $V(G)$ are vertices and elements of $E(G)$ are edges, so we can associate this straight-line drawing with the underlying abstract graph $G(V, E)$.

We will transform this definition to the following:

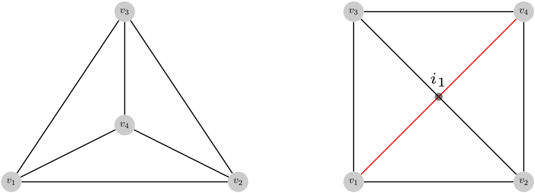
Definition 1.2 (Drawing of a graph).

A drawing D of an (undirected) graph $G(V, E)$ is an straight line embedding of G onto $\mathbb{R}^2$. The drawing can be seen as a "1-1" function $D : V \to \mathbb{R}^2$. We will write $D(G)$ to denote a drawing of graph G.

The drawing thickness

Definition 1.3 (Drawing thickness).

Let D be a drawing of $G(V, E)$. We define the drawing thickness, $\vartheta(D(G))$ to be the smallest value of k such that each edge is assigned to one of k planar layers and no two edges on the same layer cross



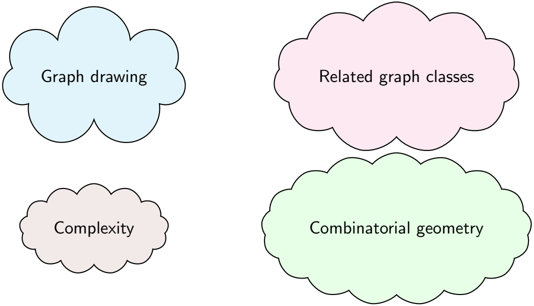
(a) $D_1(K_4)$ (b) $D_2(K_4)$

Figure: 2 different drawings of the K_4. $\vartheta(D_1(K_4)) = 1$, $\vartheta(D_2(K_4)) = 2$.

The drawing thickness

Similar ideas appear (only?) in:

- Bernhart and Kainen (1979): *"The σ-thickness $bt(G, \sigma)$ is the smallest k such that G has a k-book embedding with σ as a printed cycle"*
  printing cycle: the order of the vertices around the equivalent convex n-gon embedding on the plane
- Chung et al. (1987): *"a book embedding with specific vertex ordering"*



ATM: Flight Level organization

- ✔ Very dense traffic (lack of time & deviation alternatives)
- ✔ Sparse traffic (excess of Flight Levels available)

Or...

Joseph A. Barbetta, 1990

Outline

Outline

Graph thickness

Definition 2.1 (Graph (theoretical) thickness).

Graph-theoretical thickness, $\theta(G)$, is the minimum number of planar graphs into which a graph G can be decomposed.

- The thickness of complete graphs is known for all n:

$$\theta(K_n) = \begin{cases} 1, & 1 \leq n \leq 4 \\ 2, & 5 \leq n \leq 8 \\ 3, & 9 \leq n \leq 10 \\ \lceil \frac{n+2}{6} \rceil, & 10 < n \end{cases}$$

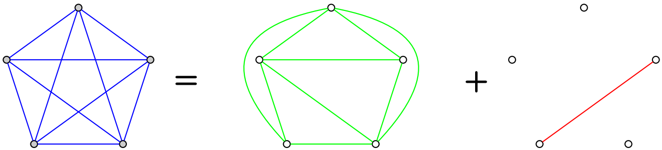


Figure: Planar decomposition of K_5: $\theta(K_5) = 2$.

Graph thickness

- $\theta(K_{m,n}) = \left\lceil \frac{mn}{2(m+n-2)} \right\rceil$, except for if mn is odd, $m > n$ and there is an even r, with $m = \left\lfloor \frac{r(n-2)}{n-r} \right\rfloor$ ([1]).

Complexity of `THICKNESS`:

Theorem 2.1.

Given a graph G, the decision problem whether G can be decomposed into 2 planar layers is NP-complete.

Proof by Mansfield ([13]) uses `PLANAR 3-SAT` (with only 3 literals(!)) as the known *NP*-complete problem for the reduction.

Graph thickness

Two equivalent ways to "see" a graph's thickness:

- Pure planar decomposition
- The "best" drawing, edges being **arbitrary curves**

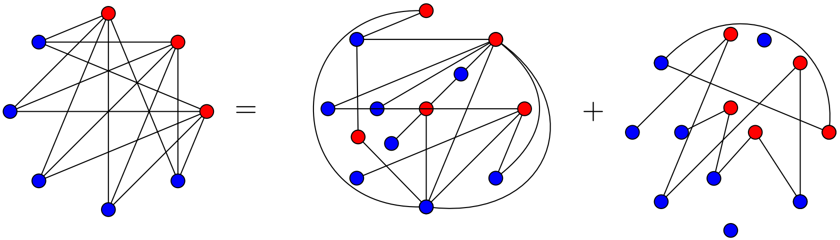


Figure: Showing (and seeing) that $\bar{\theta}(K_{3,5}) = 2$

Outline

Geometric thickness

Definition 2.2 (Geometric thickness).

We define $\bar{\theta}(G)$, the geometric thickness of a graph G, to be the smallest value of k such that we can assign planar point locations to the vertices of G, represent each edge of G as a line segment, and assign each edge to one of k layers so that no two edges on the same layer cross.

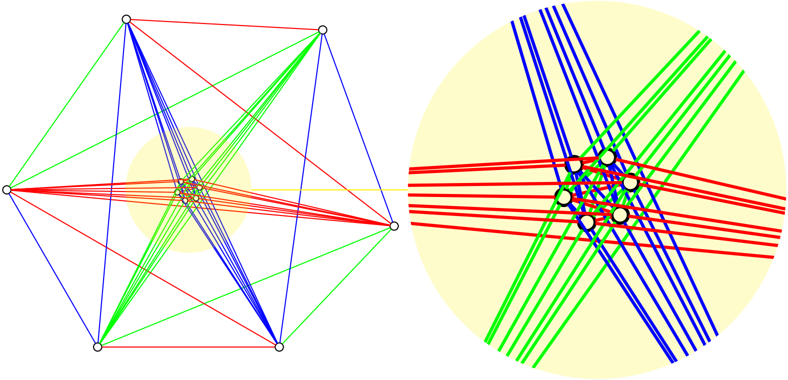
- As geometric thickness is a restriction over graph-theoretical thickness (straight line segments), it is clear that for any graph G stands $\theta(G) \leq \bar{\theta}(G)$.
- By *Fáry's theorem*, any planar graph G can be drawn in such a way that all edges are straight line segments, therefore $\bar{\theta}(G_{planar}) = 1$.
- **By definition, for any graph G and any drawing D it is true that $\bar{\theta}(G) \leq \vartheta(D(G))$.**

Geometric thickness

Theorem 2.2 (Dillencourt et al. (2000)).

For the complete K_n, $n \geq 12$ it is

$$\left\lceil \frac{n}{5.646} + 0.342 \right\rceil \leq \bar{\theta}(K_n) \leq \left\lceil \frac{n}{4} \right\rceil$$



Geometric thickness

Theorem 2.3 (Dillencourt et al. (2000)).

$$\bar{\theta}(K_n) = \begin{cases} 1, & 1 \leq n \leq 4 \\ 2, & 5 \leq n \leq 8 \\ 3, & 9 \leq n \leq 12 \\ 4, & 15 \leq n \leq 16 \end{cases}$$

For the complete bipartite graph $K_{m,n}$ it is:

$$\left\lceil \frac{mn}{2m+2n-4} \right\rceil \leq \theta(K_{m,n}) \leq \bar{\theta}(K_{m,n}) \leq \left\lceil \frac{\min(m,n)}{2} \right\rceil$$

Open Problem 1.

What is the geometric thickness of K_{13} and K_{14}? (3 or 4?)

Thickness vs. geometric thickness

- We know that $K_{6,8}$ has graph-theoretical thickness 2, but geometric thickness 3.
- Ratio between book thickness and geometric thickness has been proven unbounded by any constant factor:
- D. Eppstein ([8]) used lemmata from Ramsey theory to prove there are graphs with thickness 3 and arbitrarily large geometric thickness.
- Same problem for graphs with $\theta = 2$ remains open.

Geometric thickness

Recent result:

Theorem 2.4 (Durocher et al. (2013)).

Recognizing geometric thickness 2 graphs is NP-hard.

We may refer to the problem as `GEOM.THICKNESS`

Open Problem 2.

For a graph G, does the decision problem $\bar{\theta}(G) \leq 2$ belong to class NP?

Outline

Introduction - Motivation - Discussion

Variants of thicknesses

- Thickness
- Geometric thickness
- Book thickness

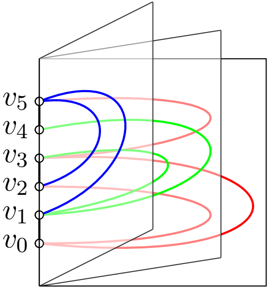
Bounds

Complexity

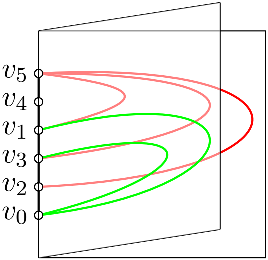
Related problems & future work

Definition 2.3 (Book embedding (L. T. Ollman,1973)).

A k-book embedding β of $G(V, E)$ is a placing of all $v \in V$ along the spine L of a book B, and a drawing of all edges $e \in E$ as arbitrary open (Jordan) arcs joining respective vertices, either in L or onto one exactly of k book pages $\{P_1, ..., P_k\}$, such that arcs on the same page do not cross.



(a) A book embedding β of G with 3 pages



(b) A book embedding β_{opt} of G with the optimum of 2 pages

Book embeddings and thickness

Naturally we will define:

Definition 2.4 (Book thickness).

We define $bt(G)$, the book thickness of a graph G, to be the smallest value of k such that G has a k-book embedding.

Book thickness alternative definition

Definition 2.5 (Book thickness via convex graph drawing).

If G has a connected component which is not a path, we can define $bt(G)$ as the smallest value of k such that vertices of G are placed in convex position, each edge of G is a line segment, and each edge is assigned to one of k layers so that no two edges on the same layer cross.

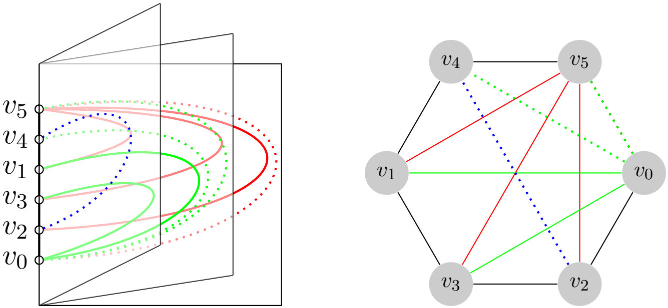


Figure: Book embedding and convex embedding.

Convex graph drawing

Definition 2.6.

A drawing D of a graph $G(V, E)$ is said to be convex if D maps set V to a convex point set on $\mathbb{R}^2$.

We will often use the notation D_{conv} to distinguish these cases. Analogously to linking geometric thickness with our drawing thickness, we have:

- $\boldsymbol{bt(G) \leq \vartheta(D_{conv}(G))}$

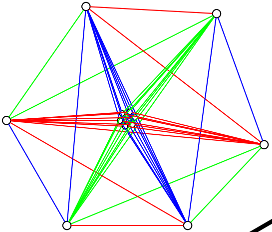
Outline

Bounds of drawing thickness

Geometrical thickness ($\bar{\theta}$)

Dillencourt et al. (2000)

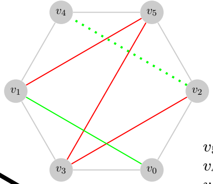
✚ only straight lines

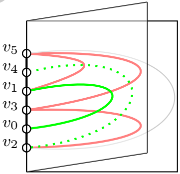


Book thickness (bt)

Bernhart and Kainen (1979)

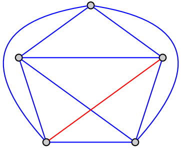
✚ convex positioning of nodes



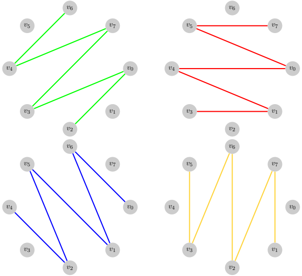


Thickness (θ)

Tutte (1963), "classical" planar decomposition



Bounds of drawing thickness

	Geometrical thickness ($\bar{\theta}$) (✚ only straight lines) $\Downarrow$ Arbitrary drawing case	Book thickness (bt) (✚ convex positioning of nodes) $\Downarrow$ Convex drawing case
Lower	$\boxed{\bar{\theta}(G) \leq \vartheta(D(G))}$	$\boxed{bt(G) \leq \vartheta(D_{conv}(G))}$
Upper		$\boxed{\vartheta(D_{conv}(G)) \leq \left\lceil \frac{n}{2} \right\rceil}$ 

That is the question

Open Problem 3 (as stated by D. Wood).

What is the minimum number of colours such that every complete geometric graph on n vertices has an edge colouring such that crossing edges get distinct colours

Open Problem 3 ("Translation").

Let the quantity $\vartheta(D(G)), |V| = n$ be bound by quantity $A(n)$, for any G of size n and drawing D. What is $A(n)$?

- The convex case dictates: $A(n) \geq \left\lceil \frac{n}{2} \right\rceil$
- Easy to see that $A(n) \leq n - 1$
- Bose et al. (2006) impoved the upper bound to $n - \sqrt{\frac{n}{12}}$

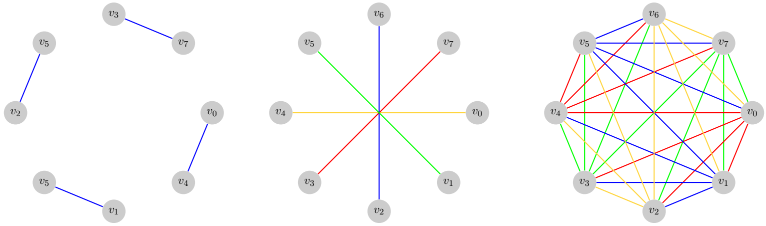
A peculiar observation

- Dillencourt et al. (2000) proved (roughly) that $\bar{\theta}(K_n) \leq \lceil n/4 \rceil$. Along with having $bt(G) = \lceil n/2 \rceil$ we may ask:

Is the convex case the worst case for our drawing thickness?

Then it would be $A(n) = \lceil n/2 \rceil$ and tight.

Sparse vs. dense graphs' drawings



(a) $\vartheta(D_{adj}(G^8_{pair})) = 1$ (b) $\vartheta(D_{opp}(G^8_{pair})) = 4$ (c) $\vartheta(D_{conv}(K_8)) = 4$

Lemma 3.1.

Let $G(V, E)$ be drawn onto $\mathbb{R}^2$ via D. It is
$\bar{\theta}(G) \leq \vartheta(D(G)) \leq \min\Big(|E|, A(n)\Big)$ *for any D.*

If it is indeed $A(n) = \lceil n/2 \rceil$ then what would be more interesting is when $\vartheta(D(G)) < \left\lceil \frac{n}{2} \right\rceil$

Outline

Determining $\vartheta(D(G))$ is NP-complete

What we will use:

- Ehrlich et al. (1976), Eppstein (2003): *Given a set of line segments on the plane, it is NP-complete to determine if the* **intersection graph** *of its edges is 3-colorable. In other words, 3−*`COLOR` *is NP-complete in SEG graphs*
- Garey et al. (1980): `COLOR` *in CIRCLE graphs is NP-complete*
- CIRCLE `3-COLOR`: is stated as *polynomially solvable* in www.graphclasses.org with Garey et al. (1980) as a reference.(**?**)

Arbitrary drawing case		Convex drawing case
$\Downarrow$		$\Downarrow$
`D.THICK`		`conv-D.THICK`
$\Downarrow$		$\Downarrow$
SEG graphs	$\supset$	CIRCLE graphs

Intersection and crossing graphs

Definition 4.1 (Intersection model (graph)).

Let $S = \{s_1, ..., s_n\}$ *be a family of line segments on the plane. Its intersection model is the graph* $H(V, E)$ *with* $V = \{s_1, ..., s_n\}$ *and* $s_i s_j \in E \Leftrightarrow s_i$ **intersects** s_j. *We will denote here* $H = I^S$. *And by definition* $H \in$ *SEG.*

Definition 4.2 (Crossing model (graph)).

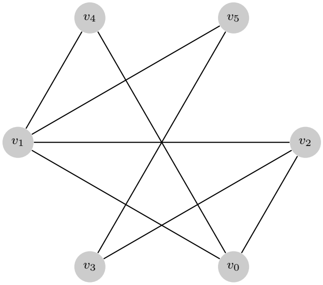
Let $S = \{s_1, ..., s_n\}$ *be a family of line segments on the plane. The crossing graph of* S *is the graph* $H(V, E)$ *with* $V = \{s_1, ..., s_n\}$ *and* $s_i s_j \in E \Leftrightarrow s_i$ **crosses** s_j. *We will denote* $H = C^S$.

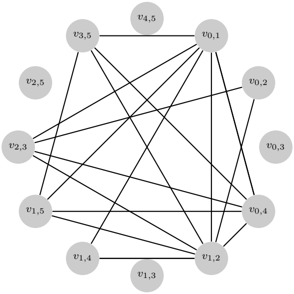
- Obviously, there are many sets S such that $C^S \neq I^S$.
- **So, if we consider a drawing of a graph, its thickness can be directy associated with the coloring of its crossing graph $C^{D(S)}$.**

CIRCLE graphs

Definition 4.3.

A graph G is a CIRCLE graph if it has an intersection model of chords of a circle.





CIRCLE graphs and convex graph drawings

Theorem 4.1.

Every convex drawing on n vertices $D_{conv}^{(n)}$ is equivalent to any other $D_{conv}^{\prime(n)}$ as long as the ordering of the vertices around the defined convex polygon remains the same, i.e. derives by rotation and refletion of the initial ordering.

Proof.

See my Diploma Thesis. □

- We can transform any convex drawing to an equivalent drawing on a circle.
- Then, drawn edges are chords of the circle.

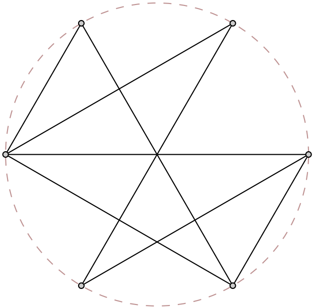
`conv-D.THICK` is NP-complete

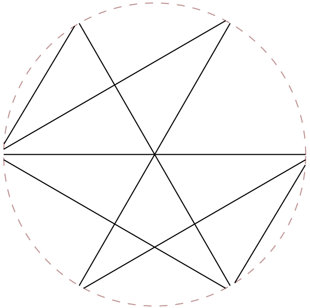
Theorem 4.2 (Chung et al. (1987)).

It is NP-complete to determine the pagenumber of a book embedding with specific vertex ordering.
Or, using our terminology:
It is NP-complete to determine the drawing thickness of a convex graph drawing.

- Chung et al.'s proof is an (easy) reduction from `CIRCLE COLOR`.
- We just note our slightly more generic class of convex drawings through the conditions of equivalence.
- Therefore, `D.THICK` is also *NP*-complete.

The proof: tweaking the endpoints





Proposition 4.1.

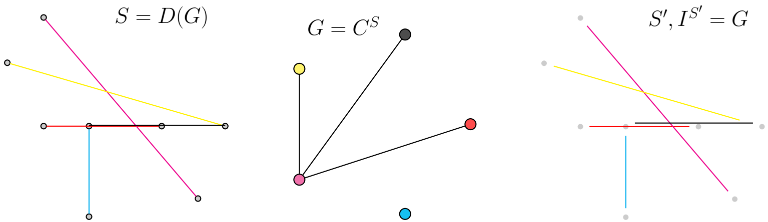
For every graph G and convex drawing D_{conv}, $C^{D_{conv}(G)} \in$ CIRCLE.

Question remains for `CIRCLE 3-COLORABILITY`

Proposition 4.2.

For every graph G and drawing D, $C^{D(G)} \in$ SEG.

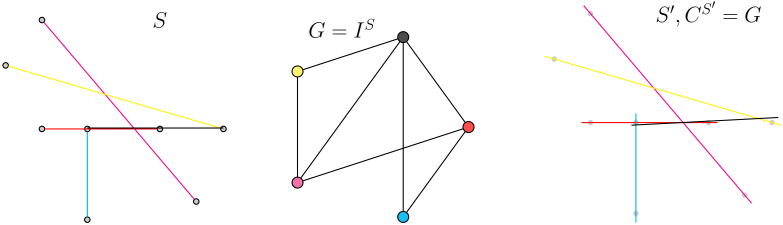
- Key: tweaking endpoints (shorten them) and splitting apart intersecting parallel segments.



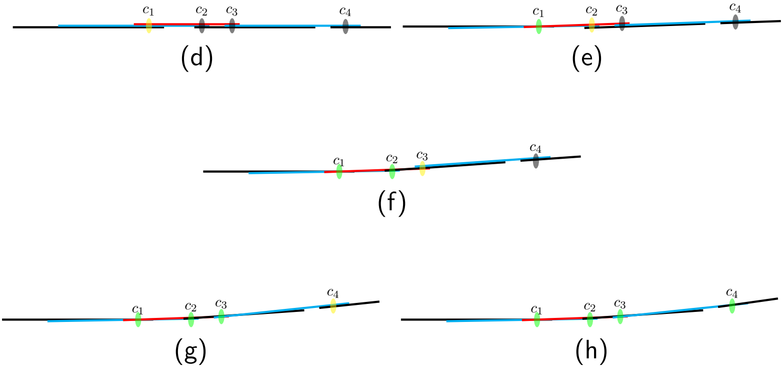
Proposition 4.3.

Let S be a set of line segments on the plane. $G = I^S \in$ SEG and we can construct in poly-time some S' such that $C^{S'} = G$.

- Key 1: tweaking endpoints (extend them)
- Key 2: **see parallel intersecting segments as an interval graph**



SEG 3-COLORABILITY $\leq^P$ 3-D.THICK



- MAX CLIQUE is polynomial time for interval graphs ([16]) and so is the problem of finding and ordering every distinct *maximal* clique, which can easily be solved in $\mathcal{O}(n)$ time using a sweep line (greedy) algorithm.

SEG 3-COLORABILITY $\leq^P$ 3-D.THICK

Theorem 4.3.

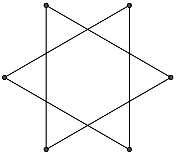
3-D.THICK is NP-complete.

- Actually, it is SEG COLORABILITY $\equiv^P$ D.THICK

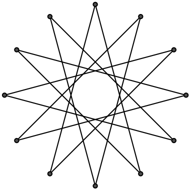
Outline

Drawing thickness of star polygons/figures

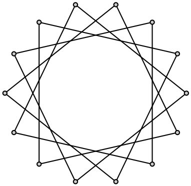
- A star polygon $\{n/k\}$, with n, k positive integers, is a figure formed by connecting with straight lines every k^{th} point out of n regularly spaced points lying on a circle.
- Originally, for a star polygon we have $\gcd(n,k) = 1$, and if $\gcd(n,k) > 1$ we often come across the term "star figure"
- It is actually convex graph drawing, according to our terminology. k is called density of the star polygon. Without loss of generality, take $k \leq \lfloor n/2 \rfloor$.



(i) $S_{6/2}$



(j) $S_{12/5}$

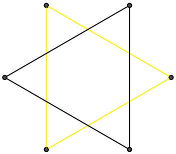


(k) $S_{14/4}$

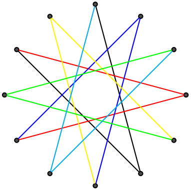
Drawing thickness of star polygons/figures

Theorem 5.1.

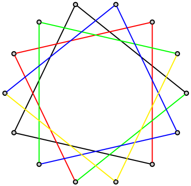
The drawing thickness of $S_{n/k}$ *is* $\vartheta(S_{n/k}) = \left\lceil \frac{n}{\lfloor \frac{n}{k} \rfloor} \right\rceil = k + \left\lceil \frac{r}{q} \right\rceil$*, the integers satisfying the Euclidean division:* $n = k \cdot q + r$, $0 \le r < k$*. In addition, for* $k_1 > k_2$ *it is* $\vartheta(S_{n/k_1}) \ge \vartheta(S_{n/k_2})$.



(l) $\vartheta(S_{6/2}) = 2$



(m) $\vartheta(S_{12/5}) = 6$

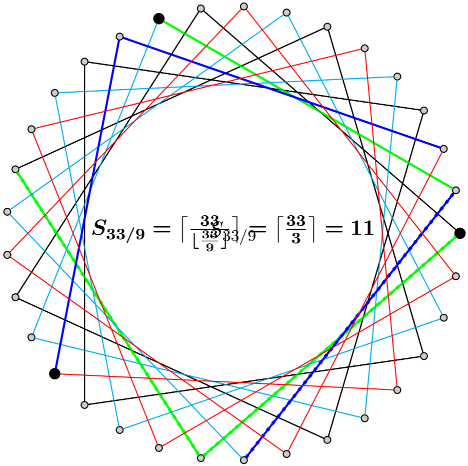


(n) $\vartheta(S_{14/4}) = 5$

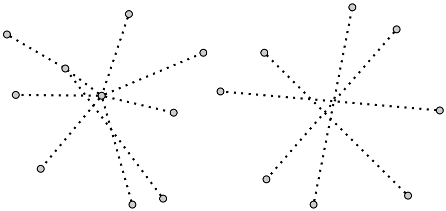
- Key for the proof is the quotient q which is the maximum number of possible edges within a single layer

Drawing thickness of star polygons/figures

- If $\gcd(n, k) = 1$, then we can draw the figure without lifting our pen and the quantity $\left\lceil \frac{n}{q} \right\rceil$ is quite evident.
- Otherwise, key #2 of the proof is the gap of size $p = \gcd(n, k)$ between the "minors" $S_{(n/p)/(k/p)}$.



Point sets that dictate $\vartheta(D(K_n)) \geq \lceil \frac{n}{2} \rceil$



A $2r$-point set P in general position on the plane is said to admit a perfect cross-matching if there are exactly r pairwise crossing segments that cover all $2r$ points.We will denote the class of such point sets by P_{pcm}.

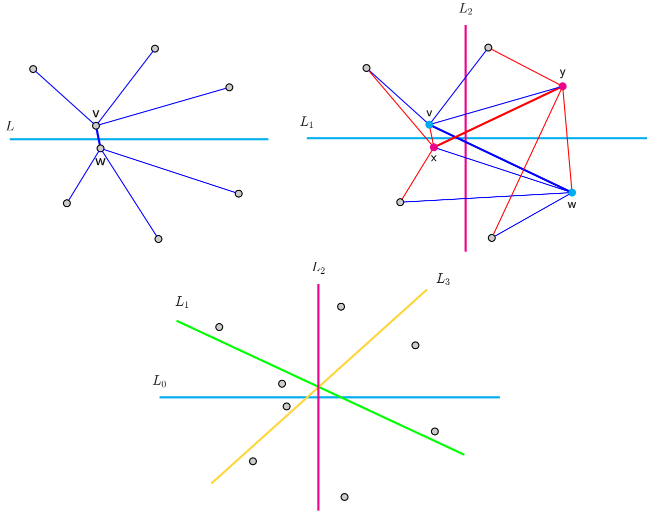
Pach and Solymosi (1999): a point set P admits a perfect cross-matching *if and only if* the number of halving lines $h(P) = r$ (in general it is $h(P) \geq n$), and there is an $O(n \log n)$-time ($O(n)$-space) algorithm that decides if $P \in P_{pcm}$.

Our interesting question was when $\vartheta(D(G)) < \lceil \frac{n}{2} \rceil$ (especially if our conjecture prooves to be correct).

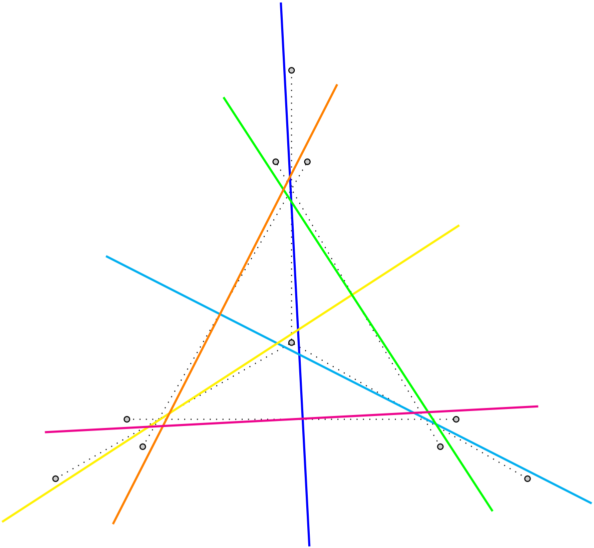
What we can answer now in polynomial time is if $D(G) \in P_{pcm}$ and thus if all edges-having lines are drawn ($O(n)$ time to check), we are sure to have $\vartheta(D(G)) = n/2$.

Point sets that allow $\vartheta(D(K_n)) \leq \lceil \frac{n}{2} \rceil$

Bose et al. (2006), using *plane spanning double stars*:



Point sets that allow $\vartheta(D(K_n)) \leq \lceil \frac{n}{2} \rceil$



Triangulation Existence problems

For the following we consider a graph $G(V, E)$ and a drawing D, and our point set is $P = D(V)$ ($|P| = n$).

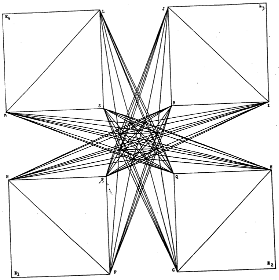
Point set triangulation (TRI): is a triangulation of the convex hull of the point set P with exactly all points of P being vertices of the triangulation. If $h(P)$ is the number of the points of P that define its convex hull, then any triangulation of P includes $e = 3n - h(P) - 3$ drawn segments/edges.

Polygon triangulation (POLY-TRI): is a decomposition of some polygon defined on P. Every triangulation of such a n-gon on the plane requires exactly $n - 3$ drawn segments/edges.

Convex triangulation (CONVEX TRI): The two definitions coincide when the point set P is convex and thus only one (convex) polygon is defined on P.

Theorem 5.2 (Lloyd (1977) and in our words).

For an arbitrary drawing D of $G(V, E)$, TRI of $P = D(V)$ is NP-complete.



CONVEX TRI is polynomially solvable $\geq_p$ CIRCLE IND. SET

IND. SET of circle graphs can be computed in polynomial time: $O(n^3)$ by Gavril (1973) and up to the most recent $O(n \min(d, \alpha))$-time output sensitive algorithm, d being the density of the graph and α being its independence number, by Nash and Gregg (2010).

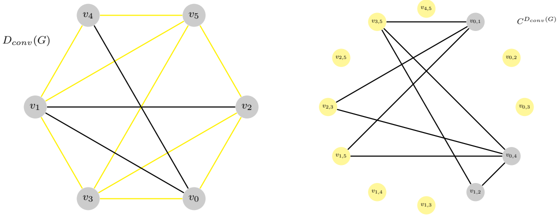


Figure: Maximal set of $9 = 2 \cdot 6 - 3$ pairwise non-crossing edges for a convex drawing and the corresponding crossing graph with max. ind. set of size 9.

What about POLY-TRI

Open Problem 4.

For given G, D, decide POLY-TRI on $P = D(V)$.

Proposition 5.1.

POLY-TRI $\in$ NP.

Proof.

Easy to see that a non-deterministic algorithm can guess some subset of E of size $2n - 3$ and check in polynomial time if the set is planar. $\square$

Some more ideas for future work

The variants of our main problem

Open Problem 5.

What is the minimum number of colours such that every complete geometric graph on n vertices has an edge colouring such that:
[Variant B] disjoint edges get distinct colours
[Variant C] non-disjoint edges get distinct colours
[Variant D] non-crossing edges get distinct colours

Little example in this direction

Variant C: non-disjoint edges get distinct colours.

- Edges with same color are a *plane matching* (at most $n/2$ edges)
- Known lower bound: $C(n) \geq n - 1$.
- Little improvement: $C(n) \geq n$.

Proof.

On the board. $\square$

Thank you!

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